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Matrix Kadomtsev–Petviashvili equation: matrix identities and explicit non-singular solutions

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Abstract

A new version of the Bäcklund–Darboux transformation for the matrix Kadomtsev–Petviashvili (KP) equation is used to construct and study explicit multi-parameter solutions and wavefunctions (in terms of the matrix exponents). A class of the self-adjoint non-singular solutions of KP I is introduced using the controllability notion from the system theory. A subclass of the rationally decaying self-adjoint non-singular solutions is studied, in particular. Several results prove new in the scalar case also.

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1. Introduction

This paper deals with a matrix analogue of the Kadomtsev–Petviashvili equation (the matrix KP equation)

$$u_t + u_{xxx} - 3(uu_x + u_x u) + \alpha^2 \omega_y = \alpha(u\omega - \omega u) \qquad \omega_x = 3u_y \tag{1.1}$$

where $u(x, t, y)$ and $\omega(x, t, y)$ are $m \times m$ matrix functions, $u_t = \frac{\partial u}{\partial t}$, and $\alpha \neq 0$ is a constant scalar. If $m = 1$, then the right-hand side of equation (1.1) turns to zero and we obtain the already classical KP equation. L, A pairs for the KP equation have been constructed in [34] and [9]. A detailed discussion of the explicit solutions can be found in [2, 19, 32]. The matrix KP equation, its integrability and solutions have been studied, for instance, in [6, 17]. The well-known Bäcklund–Darboux transformations (BDTs) are widely used to construct and study explicit solutions of the KP equation (see [1, 3, 5, 15, 18, 23, 35] and references therein). For the important modifications and generalizations of the BDT see, for instance, [8, 10, 11, 13, 18, 19, 21, 33]. (See also a recent paper [7] for results and references on the BDT and spectral theory.)

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The present paper was initiated by the growing importance of the multi-component and matrix integrable equations and their explicit solutions. An interesting class of the real non-singular and rationally decaying KP I solutions was constructed and studied in [1] (see also [30]). Here we construct and study multi-parameter explicit solutions of the matrix KP equations. A class of the rational non-singular self-adjoint (real in the scalar case) solutions is included. The corresponding wavefunctions (eigenfunctions of the matrix non-stationary Schrödinger equation) are also constructed. In particular, the formulae prove useful for the understanding of the connections between matrix identities and KP solutions actively investigated in the last few years [4, 14, 16, 31]. Several results are new in the scalar case. The constructed rational non-singular self-adjoint solutions depend generically on the polynomial in x, t, y and its conjugate, i.e. on the two real variables, although a more complicated example is treated also. The scalar solutions obtained in the interesting paper [14] are included as a subclass (see remark 2.6).

The version of the BDT that we are going to apply was initially developed in [25, 26]. We can find various applications of this method to the spectral theory and nonlinear equations in [27, 28] and [12] (see more references in these papers).

In section 2 a version of the BDT for the matrix KP equation is introduced and applied to the construction of the explicit up to the matrix exponents solutions. The self-adjoint non-singular solutions of KP I are studied in section 3. A subclass of the rationally decaying self-adjoint non-singular solutions is considered in section 4, and section 5 contains the conclusion.

2. BDT for the matrix KP

We can easily check that, supposing $u_{xy} = u_{yx}$, the matrix KP equation (1.1) is equivalent to the equations $[L_1, K_1] = 0$ and $[L_2, K_2] = 0$, where $[L, K] := LK - KL$,

$$L_1 := \frac{\partial}{\partial x} + \alpha \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix} \frac{\partial}{\partial y} - \begin{pmatrix} 0 & I_m \\ u & 0 \end{pmatrix}$$

$$K_1 := \frac{\partial}{\partial t} + 4\alpha^2 \begin{pmatrix} 0 & 0 \\ I_m & 0 \end{pmatrix} \frac{\partial^2}{\partial y^2} - 2\alpha \begin{pmatrix} 0 & 2I_m \\ u & 0 \end{pmatrix} \frac{\partial}{\partial y} + \begin{pmatrix} \alpha\omega + u_x & -2u \\ u_{xx} - \alpha u_y - 2u^2 & \alpha\omega - u_x \end{pmatrix} \tag{2.1}$$

$$L_2 := \frac{\partial^2}{\partial x^2} + \alpha \frac{\partial}{\partial y} - u \qquad K_2 := \frac{\partial}{\partial t} + 4\frac{\partial^3}{\partial x^3} - 6u\frac{\partial}{\partial x} - 3u_x + \alpha\omega \tag{2.2}$$

where I_m is the $m \times m$ identity matrix, and $L_p = L_p(\alpha, u, \omega)$, $K_p = K_p(\alpha, u, \omega)$. The auxiliary pair L_2, K_2 is traditionally used in the study of the KP equation but we need to double the order of the auxiliary systems (and use the pair L_1, K_1) to apply the approach of [25, 26, 28]. Next we introduce $N \times 2m$ matrix functions Ψ and Φ by the equations

$$L_1\Psi(x,t,y)^* = 0 \qquad K_1\Psi(x,t,y)^* = 0$$
$$L_d\Phi(x,t,y)^* = 0 \qquad K_d\Phi(x,t,y)^* = 0 \tag{2.3}$$

where the dual differential expressions L_d and K_d take the form

$$L_d = JL_1(-\alpha^*, u^*, \omega^*)J \qquad K_d = JK_1(-\alpha^*, u^*, \omega^*)J \qquad J = \mathrm{i}\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \tag{2.4}$$

and u^* denotes the conjugate transpose of u (conjugate in the scalar case). Thus, Ψ is an eigenfunction of the auxiliary and Φ is an eigenfunction of the dual systems. Similar to the binary BDT (see [1, 21] and references therein) we use both auxiliary and dual systems. We denote by $\Psi_p (p = 1, 2)$ and $\Phi_p (p = 1, 2)$ the $N \times m$ blocks of Ψ and Φ respectively, i.e. we define Ψ_p and Φ_p by the equalities $\Psi = [\Psi_1\ \Psi_2]$ and $\Phi = [\Phi_1\ \Phi_2]$. Now we introduce the $N \times N$ matrix function $S(x, t, y)$ by its derivatives

$$\begin{aligned} &S_y = \Phi\Psi^* \qquad S_x = -\alpha\Phi_2\Psi_1^* \\ &S_t = 2\alpha(2\Phi_1\Psi_2^* + \Phi_2 u\Psi_1^*) + 4\alpha^2(\Phi_{2y}\Psi_1^* - \Phi_2\Psi_{1y}^*). \end{aligned} \tag{2.5}$$

Matrix functions S, Φ and Ψ are analogues of the operators from the Lev Sakhnovich S-node [29], while the corresponding matrix identity from [28, 29] takes the form $S_y = \Phi\Psi^*$. Supposing $\Psi_{1xy}^* = \Psi_{1yx}^*$, formula (2.3) and definition (2.2) yield

$$L_2(\alpha, u, \omega)\Psi_1(x, t, y)^* = 0 \qquad K_2(\alpha, u, \omega)\Psi_1(x, t, y)^* = 0 \tag{2.6}$$

and, if u and ω satisfy matrix KP equation, then Ψ_1^* is a wavefunction of this matrix KP equation. The BDT for the non-stationary Schrödinger equation $L_2\Psi_1^* = 0$ is of interest in itself [1, 3, 24]. Furthermore, we suppose that

$$\begin{array}{lll} u_{xy} = u_{yx} & \Psi_{1xy}^* = \Psi_{1yx}^* & \Psi_{1yxx}^* = \Psi_{1xxy}^* \\ \Psi_{1xyx}^* = \Psi_{1xxy}^* & \Psi_{1xt}^* = \Psi_{1tx}^* & \Phi_{2xy} = \Phi_{2yx} \\ \Phi_{2yxx} = \Phi_{2xxy} & \Phi_{2xyx} = \Phi_{2xxy} & \Phi_{2xt} = \Phi_{2tx}. \end{array} \tag{2.7}$$

In view of equations (2.3), (2.5) and (2.6) by direct calculation using non-commutative algebra packages [22] we can obtain the result for our version of BDT (generalized BDT (GBDT) by [28]).

Theorem 2.1. *Suppose u and ω satisfy the matrix KP equation, and Ψ, Φ and S satisfy equations (2.3), (2.5) and (2.7). Then, in the points of invertibility of the S matrix functions $\widetilde{u}$ and $\widetilde{\omega}$ given by the relations*

$$\begin{aligned} &\widetilde{u}(x, t, y) := u(x, t, y) + 2\alpha X_x(x, t, y) \\ &\widetilde{\omega}(x, t, y) := \omega(x, t, y) + 6\alpha X_y(x, t, y) \\ &X(x, t, y) := \Psi_1(x, t, y)^* S(x, t, y)^{-1}\Phi_2(x, t, y) \end{aligned} \tag{2.8}$$

satisfy the matrix KP equation as well. Moreover, $\widetilde{\Psi}_1^* := \Psi_1^* S^{-1}$ is a wavefunction of this matrix KP equation, i.e., $L_2(\alpha, \widetilde{u}, \widetilde{\omega})\widetilde{\Psi}_1^* = 0$ and $K_2(\alpha, \widetilde{u}, \widetilde{\omega})\widetilde{\Psi}_1^* = 0$.

Remark 2.2. From equation (2.3) we can easily see that, if $u \equiv u^*$, $\omega \equiv \omega^*$ and $\alpha = -\alpha^*$, then we can put $\Phi = \Psi J$. In this way, setting at some fixed point $S(x_0, t_0, y_0) = S(x_0, t_0, y_0)^*$, we obtain $S(x, t, y) \equiv S(x, t, y)^*$. So according to equation (2.8) we have $\widetilde{u} \equiv \widetilde{u}^*$ and $\widetilde{\omega} \equiv \widetilde{\omega}^*$.

When $u = \omega = 0$ we can construct Ψ and Φ satisfying equation (2.3) explicitly. (The way in which operator Γ was constructed in [19] can be used, in particular.) The simplest expressions can be obtained on the matrix exponents level. In the next theorem we also construct explicitly the matrix function S.

Theorem 2.3. *Fix integers $n \geqslant N > 0$. Choose seven parameter matrices: $n \times n$ matrices A and $\widehat{A}$; $N \times n$ matrix B_1 and $n \times m$ matrix B_2; $m \times n$ matrix $\widehat{B}_1$, $n \times N$ matrix $\widehat{B}_2$, and $N \times N$ matrix C. Let R satisfy the matrix identity*

$$AR + R\widehat{A} = -\alpha B_2\widehat{B}_1. \tag{2.9}$$

Then the matrix functions

$$\Psi_1(x, t, y)^* = \widehat{B}_1 \exp(x\widehat{A} - \alpha^{-1}y\widehat{A}^2 - 4t\widehat{A}^3)\widehat{B}_2 \tag{2.10}$$

$$\Phi_2(x,t,y) = B_1 \exp(xA + \alpha^{-1}yA^2 - 4tA^3)B_2 \qquad (2.11)$$

$$\Psi_2(x,t,y) = \Psi_{1x}(x,t,y) \qquad \Phi_1(x,t,y) = -\Phi_{2x}(x,t,y) \qquad (2.12)$$

and

$$S(x,t,y) = B_1 \exp(xA + \alpha^{-1}yA^2 - 4tA^3)R\exp(x\widehat{A} - \alpha^{-1}y\widehat{A}^2 - 4t\widehat{A}^3)\widehat{B}_2 + C \qquad (2.13)$$

satisfy equations (2.3) and (2.5) and provide via equation (2.8) the explicit solutions of the matrix KP equation.

If the entries of the parameter matrices are real (and $\alpha = \pm 1$), we obtain real-valued solutions of the matrix KP II.

Remark 2.4. Under the conditions of theorem 2.3, we can assume without loss of generality that A and $\widehat{A}$ have Jordan form.

When $\alpha = \pm i$, $A = \mathrm{diag}\{\theta l_1, \theta l_2, \ldots, \theta l_n\}$, and $\widehat{A} = \mathrm{diag}\{\theta \widehat{l}_1, \theta \widehat{l}_2, \ldots, \theta \widehat{l}_n\}$ ($\theta = -\theta^*$; $l_1, \ldots, l_n, \widehat{l}_1, \ldots, \widehat{l}_n \subset \mathbb{Z}$), then $\widetilde{u}$ and $\widetilde{\omega}$ are periodic in x, t, y solutions of the matrix KP I. If $C = 0$, $\sigma(A) = \mu_0$, and $\sigma(\widehat{A}) = \widehat{\mu}_0$, where σ means spectrum, then $\widetilde{u}$ and $\widetilde{\omega}$ are rational (see corollary 4.1).

Remark 2.5. In the scalar case we have $m = 1$, and the right-hand side of equation (2.9) is of rank 1. KP equation solutions and τ-functions generated by the identities of the type (2.9) (with rank 1 right-hand side) have been considered in the interesting paper [16].

Remark 2.6. When $n = N$ and $B_1 = \widehat{B}_2 = I_n$, then according to equations (2.8) and (2.10)–(2.13) we obtain

$$X(x,t,y) = \widehat{B}_1(R + e_A(x,t,y)^{-1}Ce_{\widehat{A}}(x,t,y)^{-1})^{-1}B_2 \qquad (2.14)$$

where $e_A(x,t,y) = \exp(xA + \alpha^{-1}yA^2 - 4tA^3)$. If we assume additionally that $m = 1$, $C = 2\alpha I_n$, and matrices A and $\widehat{A}$ commute, we obtain the class of solutions introduced recently in [14] (compare equations (2.8) and (2.14) with formulae (2) and (7) in [14]).

3. Explicit self-adjoint solutions of the matrix KP I

Corollary 3.1. *If $\alpha = -\alpha^*$, $B_1 = \widehat{B}_2^*$, $\widehat{A} = A^*$, $B_2 = \mathrm{i}\widehat{B}_1^*$, $C = C^*$, and $R = R^*$, then the solutions of the matrix KP equation constructed in theorem 2.3 are self-adjoint: $\widetilde{u} = \widetilde{u}^*$ and $\widetilde{\omega} = \widetilde{\omega}^*$.*

For the case $\alpha = -\alpha^*$ we put without loss of generality $\alpha = \mathrm{i}$ and rewrite equation (1.1) in the matrix KP I form:

$$u_t + u_{xxx} - 3(uu_x + u_xu) - \omega_y = \mathrm{i}(u\omega - \omega u) \qquad \omega_x = 3u_y \qquad (3.1)$$

Under the assumptions of corollary 3.1 formulae (2.9)–(2.13) take the form

$$AR + RA^* = B_2B_2^* \qquad R = R^* \qquad (3.2)$$

$$\Phi_2(x,t,y) = \mathrm{i}\Psi_1(x,t,y) = B_1e_A(x,t,y)B_2 \qquad (3.3)$$

$$\Phi_1(x,t,y) = -\Phi_{2x}(x,t,y) = -\mathrm{i}\Psi_2(x,t,y) \qquad (3.4)$$

$$S(x,t,y) = B_1e_A(x,t,y)Re_A(x,t,y)^*B_1^* + C \qquad (3.5)$$

where $e_A(x,t,y) = \exp(xA - iyA^2 - 4tA^3)$. According to equation (2.8) the matrix KP I solutions are given by the equalities

$$\widetilde{u}(x,t,y) = 2\mathrm{i}X_x(x,t,y) \qquad \widetilde{\omega}(x,t,y) = 6\mathrm{i}X_y(x,t,y) \tag{3.6}$$

$$X(x,t,y) = \Psi_1(x,t,y)^* S(x,t,y)^{-1}\Phi_2(x,t,y). \tag{3.7}$$

In view of equations (3.2)–(3.6) without loss of generality we can assume that A has the Jordan form (recall remark 2.4).

Example 3.2. Suppose $m = 1, n = 2$, A is a Jordan cell

$$A = \begin{pmatrix} \mu_0 & 1 \\ 0 & \mu_0 \end{pmatrix} \qquad B_2 = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & c \\ c^* & d \end{pmatrix} \qquad (C = C^*). \tag{3.8}$$

By the first relation in equation (3.8) for $R = \{r_{kj}\}_{k,j=1}^2$ we obtain

$$AR + RA^* = \kappa R + \begin{pmatrix} r_{21} + r_{12} & r_{22} \\ r_{22} & 0 \end{pmatrix} \qquad \kappa := \mu_0 + \mu_0^*. \tag{3.9}$$

From the definition of A we also obtain

$$\begin{aligned} e_A(x,t,y) &= \exp\left(\mu_0 x - \mathrm{i}\mu_0^2 y - 4\mu_0^3 t\right) \\ &\quad \times \exp\left\{(A - \mu_0 I_2)\left(xI_2 - \mathrm{i}y(A + \mu_0 I_2) - 4t\left(A^2 + \mu_0 A + \mu_0^2 I_2\right)\right)\right\} \\ &= \exp\left(\mu_0 x - \mathrm{i}\mu_0^2 y - 4\mu_0^3 t\right)\left[I_2 + \begin{pmatrix} 0 & x - 2\mathrm{i}\mu_0 y - 12\mu_0^2 t \\ 0 & 0 \end{pmatrix}\right]. \end{aligned} \tag{3.10}$$

Consider the case $\kappa = \mu_0 + \mu_0^* = 0, b_1 = 1, b_2 = 0$, and $B_1 = I_2$. In view of $B_1 = I_2$ formulae (3.2)–(3.5) yield a skew-self-adjoint case of equation (2.14)

$$X(x,t,y) = \mathrm{i}B_2^*(R + e_A(x,t,y)^{-1}C(e_A(x,t,y)^{-1})^*)^{-1}B_2. \tag{3.11}$$

Taking into account equations (3.10) and (3.11) we have

$$\begin{aligned} X(x,t,y) &= \mathrm{i}B_2^*\left(R + |e(\mu_0,x,t,y)|^{-2}\left[I_2 - \begin{pmatrix} 0 & P(x,t,y) \\ 0 & 0 \end{pmatrix}\right]\right. \\ &\quad \left.\times C\left[I_2 - \begin{pmatrix} 0 & P(x,t,y) \\ 0 & 0 \end{pmatrix}\right]^*\right)^{-1} B_2 \end{aligned} \tag{3.12}$$

where

$$e(\mu_0,x,t,y) = \exp\left(\mu_0 x - \mathrm{i}\mu_0^2 y - 4\mu_0^3 t\right) \qquad P(x,t,y) = x - 2\mathrm{i}\mu_0 y - 12\mu_0^2 t. \tag{3.13}$$

(The same polynomial P has appeared already in [1].) As $\kappa = 0$, by equations (3.2) and (3.9) it follows that

$$R = \begin{pmatrix} r_0 & r_1 \\ r_1^* & 0 \end{pmatrix} \qquad r_0 = r_0^* \qquad r_1 + r_1^* = 1. \tag{3.14}$$

Notice that if $\kappa = 0$, then $|e(\mu_0,x,t,y)| = 1$. From equations (3.6), (3.12) and (3.14) by the standard calculations we now obtain

$$\begin{aligned} X(x,t,y) &= \mathrm{i}d\left(r_2 + d\left(x - 2\mathrm{i}\mu_0 y - 12\mu_0^2 t\right)\right)^{-1} \qquad (r_2 = dr_0 - |r_1 + c|^2) \\ \widetilde{u}(x,t,y) &= 2d^2\left(r_2 + d\left(x - 2\mathrm{i}\mu_0 y - 12\mu_0^2 t\right)\right)^{-2} \qquad \widetilde{\omega} = -6\mathrm{i}\mu_0\widetilde{u}. \end{aligned} \tag{3.15}$$

Suppose that $\sigma(\mathrm{i}A) \subset \mathbb{C}_+$, where σ means spectrum and $\mathbb{C}_+$ is the open upper half-plane. It is well known that in this case equation (3.2) has a unique solution:

$$R = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\lambda I_n - \mathrm{i}A)^{-1}B_2B_2^*(\lambda I_n + \mathrm{i}A^*)^{-1}\,\mathrm{d}\lambda. \tag{3.16}$$

The transformations $A \to -A, C \to -C$ yield transformations $X(x,t,y) \to -X(-x,-t,y)$, $\widetilde{u}(x,t,y) \to \widetilde{u}(-x,-t,y)$, and $\widetilde{\omega}(x,t,y) \to -\widetilde{\omega}(-x,-t,y)$. Therefore we do not need to consider the case $\sigma(\mathrm{i}A) \subset \mathbb{C}_-$ separately.

We introduce a definition from the system theory:

Definition 3.3. *A pair* A, B_2 *that satisfies the equality* $\operatorname{span}\bigcup_{l=0}^{n-1} \operatorname{Im} A^l B_2 = \mathbb{C}^n$ *(*Im *denotes image) is called full range or controllable.*

If $\sigma(\mathrm{i}A) \subset \mathbb{C}_+$ and the pair A, B_2 is full range, then according to equation (3.16) the solution of equation (3.2) is strictly positive: $R > 0$, i.e. $f^*Rf > 0$ for any $f \in \mathbb{C}^n$, $f \neq 0$. Hence in view of corollary 3.1 we obtain

Proposition 3.4. *Suppose that* $\sigma(\mathrm{i}A) \subset \mathbb{C}_+$*, the pair* A, B_2 *is full range,* $\operatorname{rank} B_1 = N$*, and* $C \geqslant 0$*. Then* $S(x,t,y)$ *given by equation (3.5) is strictly positive and therefore invertible. So the matrix KP I (3.1) solutions* $\widetilde{u}$ *and* $\widetilde{\omega}$ *given by equations (3.2)–(3.6) are non-singular.*

In the next example the solutions are non-singular.

Example 3.5. Let the parameter matrices A, B_2 and C have the form (3.8) and put $B_1 = I_2$ $(n = N = 2)$. This time we put $b_1 = 0, b_2 = 1$, and therefore the pair A, B_2 is full range. Suppose that $c = 0, d > 0$, i.e. $C \geqslant 0$, and that $\kappa = \mu_0 + \mu_0^* > 0$. So all the conditions of the proposition 3.4 are fulfilled. From equation (3.9) it follows now that

$$R = \kappa^{-1}\begin{pmatrix} 2\kappa^{-2} & -\kappa^{-1} \\ -\kappa^{-1} & 1 \end{pmatrix}. \tag{3.17}$$

We sometimes omit the variables x, t, y in our further calculations. Using equations (3.12) and (3.17) we have

$$\begin{aligned} &X = \mathrm{i}Z_1/Z_2 \qquad Z_1 = 2\kappa^{-3} + d|e(\mu_0)|^{-2}|P|^2 \\ &Z_2 = \kappa^{-4} + \kappa^{-1}d|e(\mu_0)|^{-2}(|P|^2 - \kappa^{-1}(P + P^*) + 2\kappa^{-2}). \end{aligned} \tag{3.18}$$

Taking into account equation (3.13) we easily obtain the derivatives with respect to x:

$$\begin{aligned} &Z_1' = -\kappa(Z_1 - 2\kappa^{-3}) + d|e(\mu_0)|^{-2}(P + P^*) \\ &Z_2' = -\kappa(Z_2 - \kappa^{-4}) + \kappa^{-1}d|e(\mu_0)|^{-2}(P + P^* - 2\kappa^{-1}). \end{aligned}$$

Hence we have

$$\begin{aligned} Z_1'Z_2 - Z_2'Z_1 = {} & 2\kappa^{-2}Z_2 - \kappa^{-3}Z_1 + 2\kappa^{-2}d|e(\mu_0)|^{-2}Z_1 \\ & + d|e(\mu_0)|^{-2}(P + P^*)(Z_2 - \kappa^{-1}Z_1). \end{aligned}$$

Finally in view of equations (3.6) and (3.18) we obtain

$$\begin{aligned} \widetilde{u} = {} & -\frac{2d|e(\mu_0)|^{-2}}{Z_2^2}\big(8\kappa^{-5} - 3\kappa^{-4}(P + P^*) + \kappa^{-3}(|P|^2 + 2d|e(\mu_0)|^{-2}(P + P^*)) \\ & + \kappa^{-2}d|e(\mu_0)|^{-2}(2|P|^2 - (P + P^*)^2)\big). \end{aligned} \tag{3.19}$$

The wavefunction in our case is given by the formula

$$\Psi_1^* S^{-1} = \frac{\mathrm{i}}{e(\mu_0)Z_2}(\kappa^{-2} + d|e(\mu_0)|^{-2}P^* \quad 2\kappa^{-3} - \kappa^{-2}P). \tag{3.20}$$

Notice that the second entry of $\Psi_1^* S^{-1}$ in equation (3.20) decays exponentially when $|x| \to \infty$ while the first entry grows exponentially when $x \to -\infty$. When $C > 0$ the situation is different.

Proposition 3.6. *Let the conditions of proposition 3.4 hold, and suppose that $C > 0$. Then the columns of the KP I wavefunction $\widetilde{\Psi}_1^* = \Psi_1^* S^{-1}$ are square summable in x (belong to $L^2(-\infty, \infty)$).*

Proof. According to equations (2.5) and (3.3) we have $S_x = \Psi_1 \Psi_1^*$. Therefore we obtain

$$\int_{-l}^{l} (S(x,t,y)^{-1})\Psi_1(x,t,y)\Psi_1(x,t,y)^* S(x,t,y)^{-1}\,\mathrm{d}x$$
$$= (S(-l,t,y)^{-1} - S(l,t,y)^{-1}) \leqslant C^{-1}.$$

The statement of the proposition is immediate. □

4. Rational solutions of the matrix KP

When $C = 0$, a corollary of theorem 2.3 and proposition 3.4 follows:

Corollary 4.1.

(i) Let R, Ψ_1, Φ_2, and S be defined by equations (2.9)–(2.11) and (2.13), where $\sigma(A) = \mu_0$, $\sigma(\widehat{A}) = \widehat{\mu}_0$, and $C = 0$. Then the matrix KP equation solutions $\widetilde{u}$ and $\widetilde{\omega}$ are rational in x, t, y.

(ii) Let R, Ψ_1, Φ_2, and S be defined by equations (3.2), (3.3) and (3.5). Suppose that $\sigma(\mathrm{i}A) = \lambda_0 \in \mathbb{C}_+$, the pair A, B_2 is full range, $\mathrm{rank} B_1 = N$ $(N < n)$, and $C = 0$. Then the KP I solutions $\widetilde{u}$ and $\widetilde{\omega}$ given by equations (3.6) and (3.7) are self-adjoint, non-singular, and rational in x, t, and y. Moreover, the wavefunction $\Psi_1^* S^{-1}$ is the product of the scalar multiple $e(-\mathrm{i}\lambda_0, x, t, y)$ defined in equation (3.13) and the rational multiple.

Proof. First we prove statement (i). Notice that

$$e(\mu_0, x, t, y)^{-1} e_A(x, t, y) = \exp\{A_1(\mu_0)x + A_2(\mu_0)y + A_3(\mu_0)t\} \tag{4.1}$$

where (compare with the first equality in equation (3.10))

$$A_1(\mu_0) = A - \mu_0 I_n \qquad A_2(\mu_0) = -\mathrm{i}(A^2 - \mu_0^2 I_n) \qquad A_3(\mu_0) = -4\left(A^3 - \mu_0^3 I_n\right)$$

i.e., A_1, A_2, and A_3 are nilpotent matrices. (Recall that A is called nilpotent if $A^k = 0$ for some $k \geqslant 0$.) In the same way we obtain

$$e(\widehat{\mu}_0, x, t, y)^{-1} e_{\widehat{A}}(x, t, y) = \exp\{\widehat{A}_1(\widehat{\mu}_0)x + \widehat{A}_2(\widehat{\mu}_0)y + \widehat{A}_3(\widehat{\mu}_0)t\} \tag{4.2}$$

where $\widehat{A}_1$, $\widehat{A}_2$ and $\widehat{A}_3$ are nilpotent. Therefore, the right-hand sides of equations (4.1) and (4.2) are rational. Hence, taking into account that $C = 0$ we derive from equations (2.10), (2.11), (2.13) and (2.8) that X is rational. Using equation (2.8) the matrix functions $\widetilde{u}$ and $\widetilde{\omega}$ are rational as well.

Let us now prove part (ii) of the corollary. The rationality of $\widetilde{u}$ and $\widetilde{\omega}$ follows from part (i). The self-adjointness and regularity of the solutions follows from corollary 3.1 and proposition 3.4, respectively. Finally, according to equations (3.3), (3.5) and (4.1), $e(-\mathrm{i}\lambda_0)^{-1}\Psi_1^* S^{-1}$ is rational. □

Furthermore, we study the rational non-singular solutions described in part (ii) of the corollary. In the scalar case, this type of multi-lump solution has been considered in [1]. The problem of the existence of slowly decaying non-singular solutions in the case of one space variable is of interest also (see the discussion in [20] and references therein). Notice that if $C = 0$ and $n = N$ we obtain only trivial solutions.

Example 4.2. Let A and B_2 be given by equation (3.8), where $b_1 = 0, b_2 = 1$, and $\kappa > 0$ again. Put $C = 0$, $N = 1$, and $B_1 = (1 \quad 1)$. In this way, all the conditions of corollary 4.1 part (ii) are fulfilled. From equations (3.5) and (3.10) we obtain

$$S = |e(\mu_0)|^2 B_1 \left[I_2 + \begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix} \right] R \left[I_2 + \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} \right]^* B_1^*. \tag{4.3}$$

As A and B_2 coincide with the corresponding matrices from example 3.5, the matrix R is given by equation (3.17). Using equations (3.17) and (4.3) we easily calculate

$$S = \kappa^{-1}|e(\mu_0)|^2(2\kappa^{-2} - \kappa^{-1}(P + P^* + 2) + |P + 1|^2). \tag{4.4}$$

From equations (3.3), (3.7) and (4.4) we derive

$$X = \frac{\mathrm{i}\kappa|P + 1|^2}{2\kappa^{-2} - \kappa^{-1}(P + P^* + 2) + |P + 1|^2}. \tag{4.5}$$

Finally the lump solution $\widetilde{u}$ takes the form

$$\widetilde{u} = \frac{2[(P + 1)^2 + (P^* + 1)^2 - 2\kappa^{-1}(P + P^* + 2)]}{(2\kappa^{-2} - \kappa^{-1}(P + P^* + 2) + |P + 1|^2)^2}. \tag{4.6}$$

Consider now a simple matrix lump solution.

Example 4.3. In this example we take the same matrices A, B_1 and $C = 0$ as in the previous example 4.2 and put $m = 2$, $B_2 = I_2$. According to equations (3.2) and (3.9) the entries r_{12}, r_{21} and r_{22} of R coincide with the corresponding entries in the previous example (see equation (3.17)) but we have $r_{11} = \kappa^{-1} + 2\kappa^{-3}$. Therefore, equation (4.3) now yields

$$S = \kappa^{-1}|e(\mu_0)|^2(1 + 2\kappa^{-2} - \kappa^{-1}(P + P^* + 2) + |P + 1|^2). \tag{4.7}$$

Analogously to example 4.2 we obtain

$$X = \frac{\mathrm{i}\kappa}{1 + 2\kappa^{-2} - \kappa^{-1}(P + P^* + 2) + |P + 1|^2} \begin{pmatrix} P^* + 1 \\ 1 \end{pmatrix} (1 \quad P + 1). \tag{4.8}$$

Recall that $\widetilde{u} = 2\mathrm{i}X_x, \widetilde{\omega} = 6\mathrm{i}X_y$.

It has already been mentioned that without loss of generality we can assume that A has the Jordan form. Therefore, under the conditions of corollary 4.1 the nontrivial generic case of the rational multi-lump solutions corresponds to the block diagonal $2l \times 2l$ matrix A with the 2×2 Jordan cells:

$$A = A(\mu_0) = \mathrm{diag}\{a, \ldots, a\} \qquad a = \begin{pmatrix} \mu_0 & 1 \\ 0 & \mu_0 \end{pmatrix} \quad (\kappa = \mu_0 + \mu_0^* > 0). \tag{4.9}$$

Put $D(P) = I_{2l} + PA(0)$. The matrix function X now takes the form

$$X = \mathrm{i}B_2^* D(P)^* B_1^* (B_1 D(P) R D(P)^* B_1^*)^{-1} B_1 D(P) B_2. \tag{4.10}$$

Remark 4.4. In the generic case (4.10) as well as in the examples (4.5) and (4.8) we have $X(x, t, y) = \widetilde{X}(P, P^*)$, where $P(x, t, y) = x - 2\mathrm{i}\mu_0 y - 12\mu_0^2 t$. (Up to a constant P coincides with f in [1].) Thus we obtain

$$\begin{aligned} \widetilde{u}(x, t, y) &= 2\mathrm{i}(\widetilde{X}_P(P, P^*) + \widetilde{X}_{P^*}(P, P^*)) \\ \widetilde{\omega}(x, t, y) &= 12(\mu_0 \widetilde{X}_P(P, P^*) - \mu_0^* \widetilde{X}_{P^*}(P, P^*)). \end{aligned} \tag{4.11}$$

In other words $\widetilde{u}$ and $\widetilde{\omega}$ depend on the two real variables: real and imaginary parts of P.

The asymptotics of the generic solutions follows from equation (4.11) and the next proposition is easily derived from equation (4.10).

Proposition 4.5. *Suppose* det $B_1A(0)RA(0)^*B_1^* \neq 0$. *Then the asymptotics of the derivatives of the matrix function* $\widetilde{X}$ *defined in remark 4.4 is given by*

$$\widetilde{X}_P(P, P^*) = \frac{\mathrm{i}}{P^2}K_1^*K_2^{-1}\big(K_3K_2^{-1}K_1 - B_1B_2\big) + O\left(\frac{1}{|P|^3}\right) \qquad (|P| \to \infty) \tag{4.12}$$

$$\widetilde{X}_{P^*}(P, P^*) = \frac{\mathrm{i}}{(P^*)^2}\big(K_1^*K_2^{-1}K_3^* - B_2^*B_1^*\big)K_2^{-1}K_1 + O\left(\frac{1}{|P|^3}\right) \qquad (|P| \to \infty) \tag{4.13}$$

where

$$K_1 = B_1A(0)B_2 \qquad K_2 = B_1A(0)RA(0)^*B_1^* \qquad K_3 = B_1RA(0)^*B_1^*.$$

If A contains 3×3 Jordan cells, then we can no longer present $\widetilde{u}$ as a function of P and P^*.

Example 4.6. Suppose $m = 2$, $N = 1$, $n = 3$, $B_1 = (1 \quad 0 \quad 0)$, A is a 3×3 Jordan cell:

$$A = \begin{pmatrix} \mu_0 & 1 & 0 \\ 0 & \mu_0 & 1 \\ 0 & 0 & \mu_0 \end{pmatrix} \qquad (\kappa = \mu_0 + \mu_0^* > 0) \qquad B_2 = \begin{pmatrix} 0 \\ I_2 \end{pmatrix}. \tag{4.14}$$

Similar to equation (3.10) we obtain

$$\begin{aligned} e_A(x, t, y) &= e(\mu_0)\exp\left\{(A - \mu_0I_3)\left(xI_3 - \mathrm{i}y(A + \mu_0I_3) - 4t\left(A^2 + \mu_0A + \mu_0^2I_3\right)\right)\right\} \\ &= \mathrm{e}(\mu_0)\left[I_3 + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\left(xI_3 - \mathrm{i}y(A + \mu_0I_3) - 4t\left(A^2 + \mu_0A + \mu_0^2I_3\right)\right)\right. \\ &\quad \left. + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\left(xI_3 - \mathrm{i}y(A + \mu_0I_3) - 4t\left(A^2 + \mu_0A + \mu_0^2I_3\right)\right)^2\right]. \end{aligned}$$

Simple calculations now yield

$$e_A = e(\mu_0)\begin{pmatrix} 1 & P & Q \\ 0 & 1 & P \\ 0 & 0 & 1 \end{pmatrix} \tag{4.15}$$

where $Q(x, t, y) = \frac{1}{2}P(x, t, y)^2 - \mathrm{i}y - 12\mu_0t$. From equations (3.3) and (4.15) it follows that

$$B_1e_A = e(\mu_0)(1 \quad P \quad Q) \qquad \Phi_2 = \mathrm{i}\Psi_1 = e(\mu_0)(P \quad Q). \tag{4.16}$$

Using equations (3.5) and (4.16) we obtain

$$S = |e(\mu_0)|^2(1 \quad P \quad Q)R(1 \quad P \quad Q)^*. \tag{4.17}$$

Finally according to equations (4.16) and (4.17) we obtain

$$X = \frac{\mathrm{i}}{(1 \quad P \quad Q)R(1 \quad P \quad Q)^*}\begin{pmatrix} P^* \\ Q^* \end{pmatrix}(P \quad Q). \tag{4.18}$$

To construct R in equation (4.18) we derive from equation (3.2) and a relation similar to equation (3.9) the equality

$$\kappa R + \begin{pmatrix} r_{21} + r_{12} & r_{22} + r_{13} & r_{23} \\ r_{31} + r_{22} & r_{32} + r_{23} & r_{33} \\ r_{32} & r_{33} & 0 \end{pmatrix} = \mathrm{diag}\{0, 1, 1\}. \tag{4.19}$$

Using equation (4.19) we have $r_{33} = \kappa^{-1}, r_{23} = r_{32} = -\kappa^{-2}, r_{31} = r_{13} = \kappa^{-3}, r_{22} = 2\kappa^{-3} + \kappa^{-1}, r_{21} = r_{12} = -\kappa^{-2} - 3\kappa^{-4}, r_{11} = 2(\kappa^{-3} + 3\kappa^{-5})$. Now formulae (3.6) and (4.18) express $\widetilde{u}$ and $\widetilde{\omega}$ via polynomials P and Q.

5. Conclusion

Thus, the GBDT and various matrix and operator identities prove fruitful for the construction of the matrix KP equation solutions. A notion from the system theory has been used for the construction of the non-singular solutions. (See also the representation of the Darboux matrix in the form of the transfer matrix function from the system theory in [25–28].) The solutions obtained in this way can be treated as a type of soliton–multi-lump interaction, and a subclass of the slowly decaying self-adjoint non-singular rational matrix KP I solutions is included. This subclass corresponds to the parameter matrix A with a single eigenvalue. Some developments of the important paper [1] have been achieved. The generic case of the block diagonal A consisting of the 2×2 Jordan cells has been studied. In this case $\widetilde{u}$ and $\widetilde{\omega}$ depend on the polynomials P and P^* $\left(P(x,t,y) = x - 2\mathrm{i}\mu_0 y - 12\mu_0^2 t\right)$, and the asymptotics of $\widetilde{u}$ and $\widetilde{\omega}$ when $|P| \to \infty$ is described. Finally, an example of the 3×3 Jordan cell A has been treated.

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